MATH 1700: SECTION 12.4: THE DOT PRODUCT

In the previous section, we learned how add and subtract vectors and how to multiply vectors by scalars. In this section, we define one way to multiply two vectors.

THE DOT PRODUCT: Given vectors $\vec{v} = \langle v_1, v_2 \rangle$ and $\vec{w} = \langle w_1, w_2 \rangle$, the **dot product** of $\vec{v}$ and $\vec{w}$ is given by

$$\vec{v} \cdot \vec{w} = \langle v_1, v_2 \rangle \cdot \langle w_1, w_2 \rangle = v_1 w_1 + v_2 w_2$$

For example, if $\vec{v} = \langle 3, 4 \rangle$ and $\vec{w} = \langle 1, -2 \rangle$,then $\vec{v} \cdot \vec{w} = \langle 3, 4 \rangle \cdot \langle 1, -2 \rangle = (3)(1) + (4)(-2) = -5$.

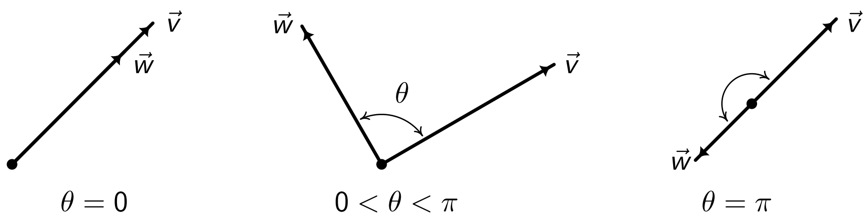
Note that the dot product takes two *vectors* and produces a *scalar*. For that reason, the quantity $\vec{v} \cdot \vec{w}$ is often called the **scalar product** of $\vec{v}$ and $\vec{w}$. The dot product enjoys the following properties.

PROPERTIES OF THE DOT (SCALAR) PRODUCT:

- **COMMUTATIVE:** For all vectors $\vec{v}$ and $\vec{w}$, $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.
- **DISTRIBUTIVE PROPERTY OVER VECTOR ADDITION:**
  For all vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$.
- **SCALAR PROPERTY:** For all vectors $\vec{v}$ and $\vec{w}$ and scalars k, $(k\vec{v}) \cdot \vec{w} = k(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (k\vec{w})$.
- **MAGNITUDE:** For all vectors $\vec{v}$, $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$.

EXAMPLE 1: Prove the identity: $\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 - 2(\vec{v} \cdot \vec{w}) + \|\vec{w}\|^2$.

Suppose $\vec{v}$ and $\vec{w}$ are two nonzero vectors. If we draw $\vec{v}$ and $\vec{w}$ with the same initial point, we define the **angle between** $\vec{v}$ and $\vec{w}$ to be the angle θ, $0 \leq \theta \leq \pi$, determined by the rays containing the vectors $\vec{v}$ and $\vec{w}$:



GEOMETRIC INTERPRETATION OF THE DOT PRODUCT:

If $\vec{v}$ and $\vec{w}$ are nonzero vectors and θ is the angle between them, then $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos(\theta)$.

If $0 < \theta < \pi$, the vectors $\vec{v}$, $\vec{w}$ and $\vec{v} - \vec{w}$ determine a triangle with side lengths $\|\vec{v}\|$, $\|\vec{w}\|$ and $\|\vec{v} - \vec{w}\|$, respectively, as seen in the diagram below.

The Law of Cosines yields $\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\| \cos(\theta)$.

From Example 1, we also know that $\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 - 2(\vec{v} \cdot \vec{w}) + \|\vec{w}\|^2$.

Setting these two expressions for $\|\vec{v} - \vec{w}\|^2$ equal to each other gives:

$$\|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\| \cos(\theta) = \|\vec{v}\|^2 - 2(\vec{v} \cdot \vec{w}) + \|\vec{w}\|^2,$$

which reduces to $-2\|\vec{v}\|\|\vec{w}\| \cos(\theta) = -2(\vec{v} \cdot \vec{w})$. Hence, $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos(\theta)$.

The cases where $\theta = 0$ or $\theta = \pi$ are addressed in the text and are good exercises to think through.

THE ANGLE BETWEEN TWO VECTORS:

Let $\vec{v}$ and $\vec{w}$ be nonzero vectors and let θ the angle between $\vec{v}$ and $\vec{w}$. Then

$$\theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right) = \arccos(\hat{v} \cdot \hat{w})$$

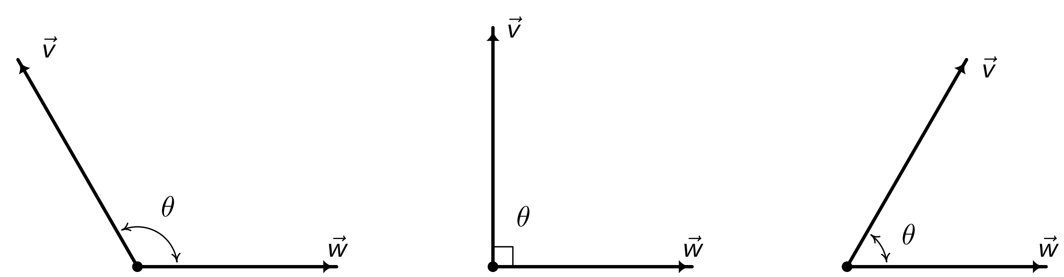
EXAMPLE 2: Find the angle between the following pairs of vectors. Graph each pair of vectors in standard position to check the reasonableness of your answer.

1. $\vec{v} = \left\langle 3, -3\sqrt{3} \right\rangle$, and $\vec{w} = \left\langle -\sqrt{3}, 1 \right\rangle$

2. $\vec{v} = \langle 2, 2 \rangle$, and $\vec{w} = \langle 5, -5 \rangle$

3. $\vec{v} = \langle 3, -4 \rangle$, and $\vec{w} = \langle 2, 1 \rangle$

GEOMETRIC SIGNIFICANCE OF THE SIGN OF THE DOT PRODUCT:



$\vec{v} \cdot \vec{w} < 0$ — θ is obtuse

$\vec{v} \cdot \vec{w} = 0$ — $\theta = \frac{\pi}{2} = 90^\circ$

$\vec{v} \cdot \vec{w} > 0$ — θ is acute

THE DOT PRODUCT DETECTS ORTHOGONALITY:

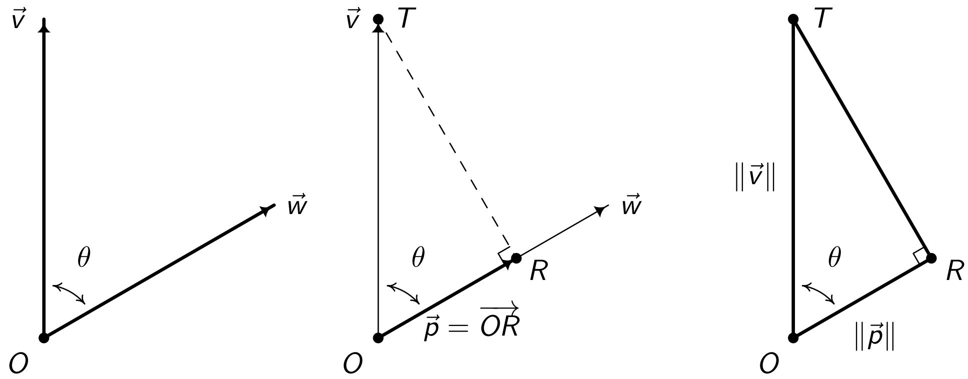
For nonzero vectors $\vec{v}$ and $\vec{w}$, $\vec{v} \perp \vec{w}$ if and only if $\vec{v} \cdot \vec{w} = 0$.

EXAMPLE 3: Let L_1 be the line $y = m_1 x + b_1$ and let L_2 be the line $y = m_2 x + b_2$.

Prove that L_1 is perpendicular to L_2 if and only if $m_1 \cdot m_2 = -1$.

VECTOR PROJECTIONS:

Consider the two nonzero vectors $\vec{v}$ and $\vec{w}$ drawn with a common initial point O below. For the moment, assume that the angle between $\vec{v}$ and $\vec{w}$, θ, is acute.



We wish to develop a formula for the vector $\vec{p}$, indicated below, which is called the **orthogonal projection of $\vec{v}$ onto $\vec{w}$**. The vector $\vec{p}$ is obtained geometrically as follows: drop a perpendicular from the terminal point T of $\vec{v}$ to the vector $\vec{w}$ and call the point of intersection R. The vector $\vec{p}$ is then defined as $\vec{p} = \overrightarrow{OR}$.

Like any vector, $\vec{p}$ is determined by its magnitude $\|\vec{p}\|$ and its direction $\hat{p}$: $\vec{p} = \|\vec{p}\|\hat{p}$. Since we want $\hat{p}$ to have the same direction as $\vec{w}$, we have $\hat{p} = \hat{w}$.

Using the right triangle $\triangle ORT$, we find $\cos(\theta) = \frac{\|\vec{p}\|}{\|\vec{v}\|}$, or, equivalently, $\|\vec{p}\| = \|\vec{v}\|\cos(\theta)$. Rewriting, we get:

$$\|\vec{p}\| = \|\vec{v}\|\cos(\theta) = \frac{\|\vec{v}\|\|\vec{w}\|\cos(\theta)}{\|\vec{w}\|} = \frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|} = \vec{v}\cdot\left(\frac{1}{\|\vec{w}\|}\vec{w}\right) = \vec{v}\cdot\hat{w}.$$

Hence, $\|\vec{p}\| = \vec{v}\cdot\hat{w}$, and since $\hat{p} = \hat{w}$, we have $\vec{p} = \|\vec{p}\|\hat{p} = (\vec{v}\cdot\hat{w})\hat{w}$.

The cases where θ is an obtuse or right angle are addressed in the text and are good exercises to think through.

VECTOR PROJECTION: Let $\vec{v}$ and $\vec{w}$ be nonzero vectors.
The **orthogonal projection of $\vec{v}$ onto $\vec{w}$**, denoted $\text{proj}_{\vec{w}}(\vec{v})$ is given by $\text{proj}_{\vec{w}}(\vec{v}) = (\vec{v}\cdot\hat{w})\hat{w}$.

FORMULAS FOR VECTOR PROJECTIONS: If $\vec{v}$ and $\vec{w}$ are nonzero vectors then

$$\text{proj}_{\vec{w}}(\vec{v}) = (\vec{v}\cdot\hat{w})\hat{w} = \left(\frac{\vec{v}\cdot\vec{w}}{\|\vec{w}\|^2}\right)\vec{w} = \left(\frac{\vec{v}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}$$

EXAMPLE 4: Let $\vec{v} = \langle 1, 8\rangle$ and $\vec{w} = \langle -1, 2\rangle$. Find $\vec{p} = \text{proj}_{\vec{w}}(\vec{v})$. Check your answer geometrically.

GENERALIZED VECTOR DECOMPOSITION THEOREM: Let $\vec{v}$ and $\vec{w}$ be nonzero vectors. There are unique vectors $\vec{p}$ and $\vec{q}$ such that $\vec{v} = \vec{p} + \vec{q}$ where $\vec{p} = k\vec{w}$ for some scalar k, and $\vec{q} \cdot \vec{w} = 0$.

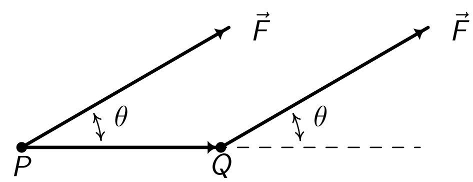
If the vectors $\vec{p}$ and $\vec{q}$ are nonzero, then we can say $\vec{p}$ is 'parallel' to $\vec{w}$ and $\vec{q}$ is 'orthogonal' to $\vec{w}$. In this case, the vector $\vec{p}$ is sometimes called the 'vector component of $\vec{v}$ *parallel* to $\vec{w}$' and $\vec{q}$ is called the 'vector component of $\vec{v}$ *orthogonal* to $\vec{w}$.'

The proof of the decomposition theorem is in the text and comprises of two parts: first, we use projections to provide vectors $\vec{p}$ and $\vec{q}$ which satisfy the properties stated in the theorem; second, we prove the vectors we find are the *only* such vectors which satisfy the properties in the theorem.

WORK:

We close this section with an application of the dot product. In Physics, if a constant force F is exerted over a distance d, the **work** W done by the force is given by $W = Fd$. Here, the assumption is that the force is being applied in the direction of the motion. If the force applied is not in the direction of the motion, we can use the dot product to find the work done.

Consider the scenario sketched below in which the constant force $\vec{F}$ is applied to move an object from the point P to the point Q. Here the force is being applied at an angle θ as opposed to being applied directly in the direction of the motion.



To find the work W done in this scenario, we need to find how much of the force $\vec{F}$ is in the direction of the motion $\overrightarrow{PQ}$. This is precisely what the dot product $\vec{F} \cdot \widehat{PQ}$ represents.

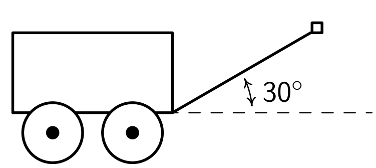
Since the distance the object travels is $\|\overrightarrow{PQ}\|$, we get $W = (\vec{F} \cdot \widehat{PQ})\|\overrightarrow{PQ}\|$. Since $\overrightarrow{PQ} = \|\overrightarrow{PQ}\|\widehat{PQ}$, we can simplify this formula as follows: $W = (\vec{F} \cdot \widehat{PQ})\|\overrightarrow{PQ}\| = \vec{F} \cdot (\|\overrightarrow{PQ}\|\widehat{PQ}) = \vec{F} \cdot \overrightarrow{PQ}$.

Hence, $W = \vec{F} \cdot \overrightarrow{PQ} = \|\vec{F}\|\|\overrightarrow{PQ}\|\cos(\theta)$, where θ is the angle between the applied force $\vec{F}$ and the trajectory of the motion $\overrightarrow{PQ}$. We have proved the following.

WORK AS A DOT PRODUCT: Suppose a constant force $\vec{F}$ is applied along the vector $\overrightarrow{PQ}$. T

The work W done by $\vec{F}$ is given by $W = \vec{F} \cdot \overrightarrow{PQ} = \|\vec{F}\|\|\overrightarrow{PQ}\|\cos(\theta)$, where θ is the angle between $\vec{F}$ and $\overrightarrow{PQ}$.

EXAMPLE 5: Taylor exerts a force of 10 pounds to pull her wagon a distance of 50 feet over level ground. If the handle of the wagon makes a 30° angle with the horizontal, how much work did Taylor do pulling the wagon? Assume the force of 10 pounds is exerted at a 30° angle for the duration of the 50 feet.



HOMEWORK: Section 12.4: Page 1081: 1 - 25 odd.